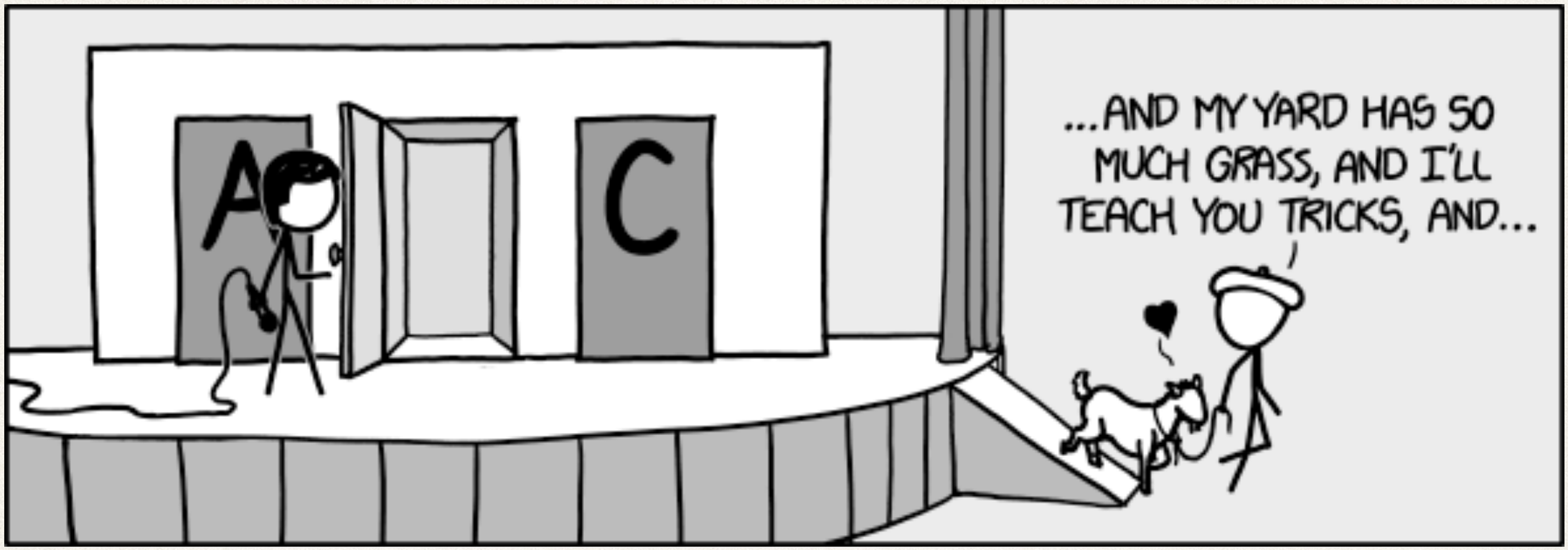
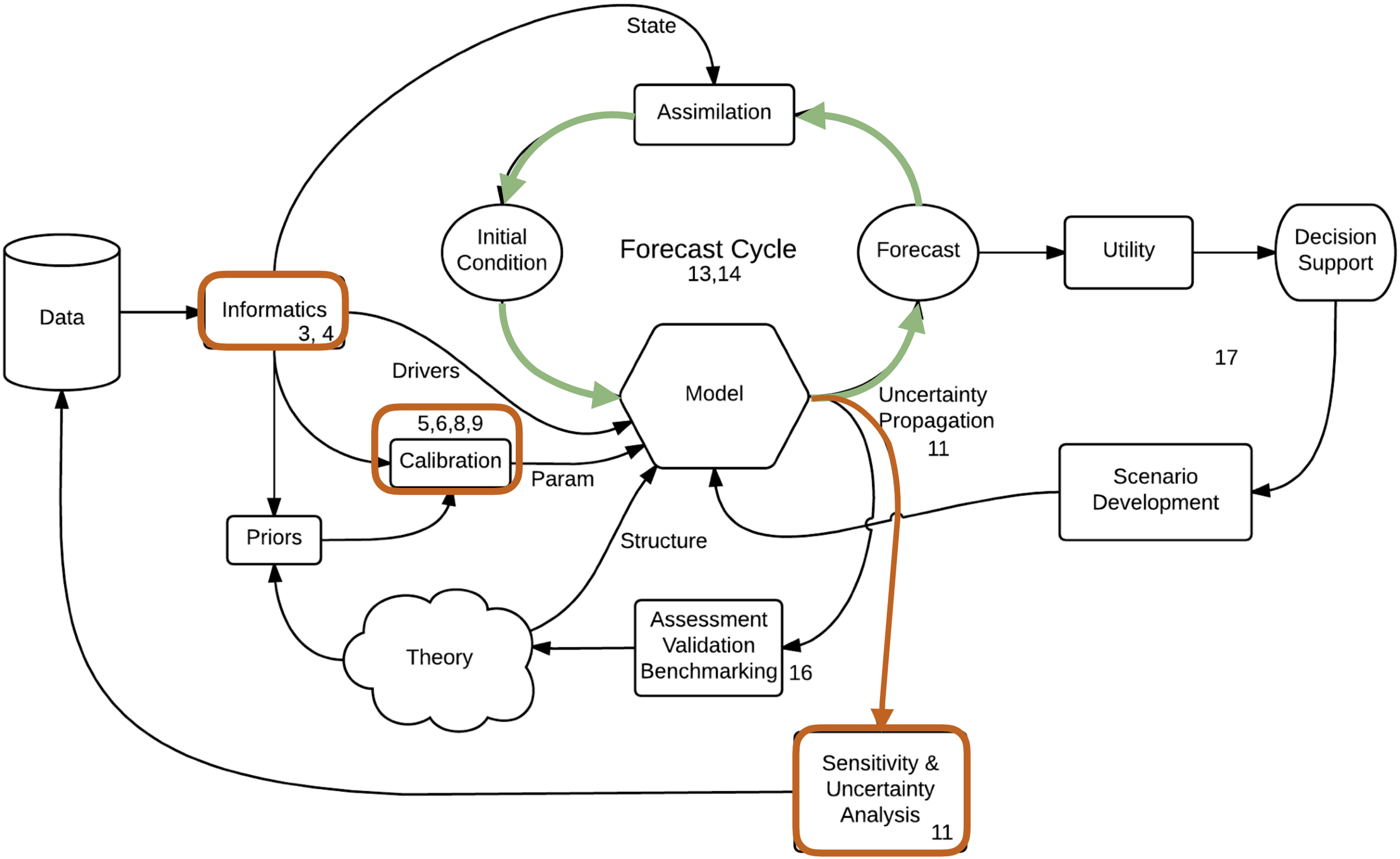


Data Assimilation 1: Analytical Methods

Lesson 9

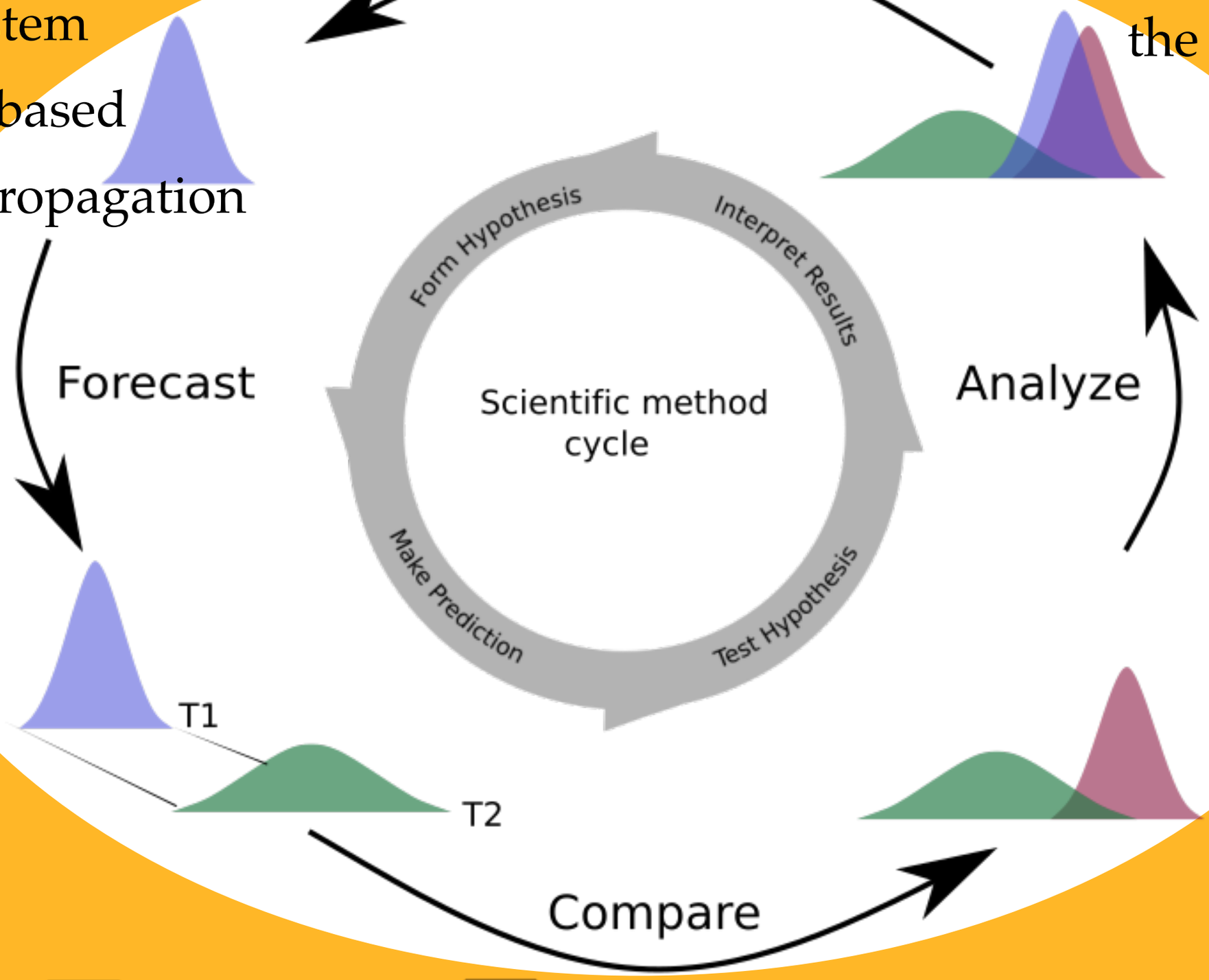

State
Assimilation
Initial Condition
Forecast Cycle
13,14
Forecast
Utility
Decision Support
Data
Informatics
3, 4
17
Drivers
Model
Uncertainty Propagation
11
5,6,8,9
Calibration
Param
Scenario Development
Priors
Structure
Theory
Assessment Validation Benchmarking
16
Sensitivity & Uncertainty Analysis
11

FORECAST-ANALYSIS CYCLE

Predict the future using your current understanding of the system

- Model-based
- Error-propagation

Update prior understanding of the system based on new information



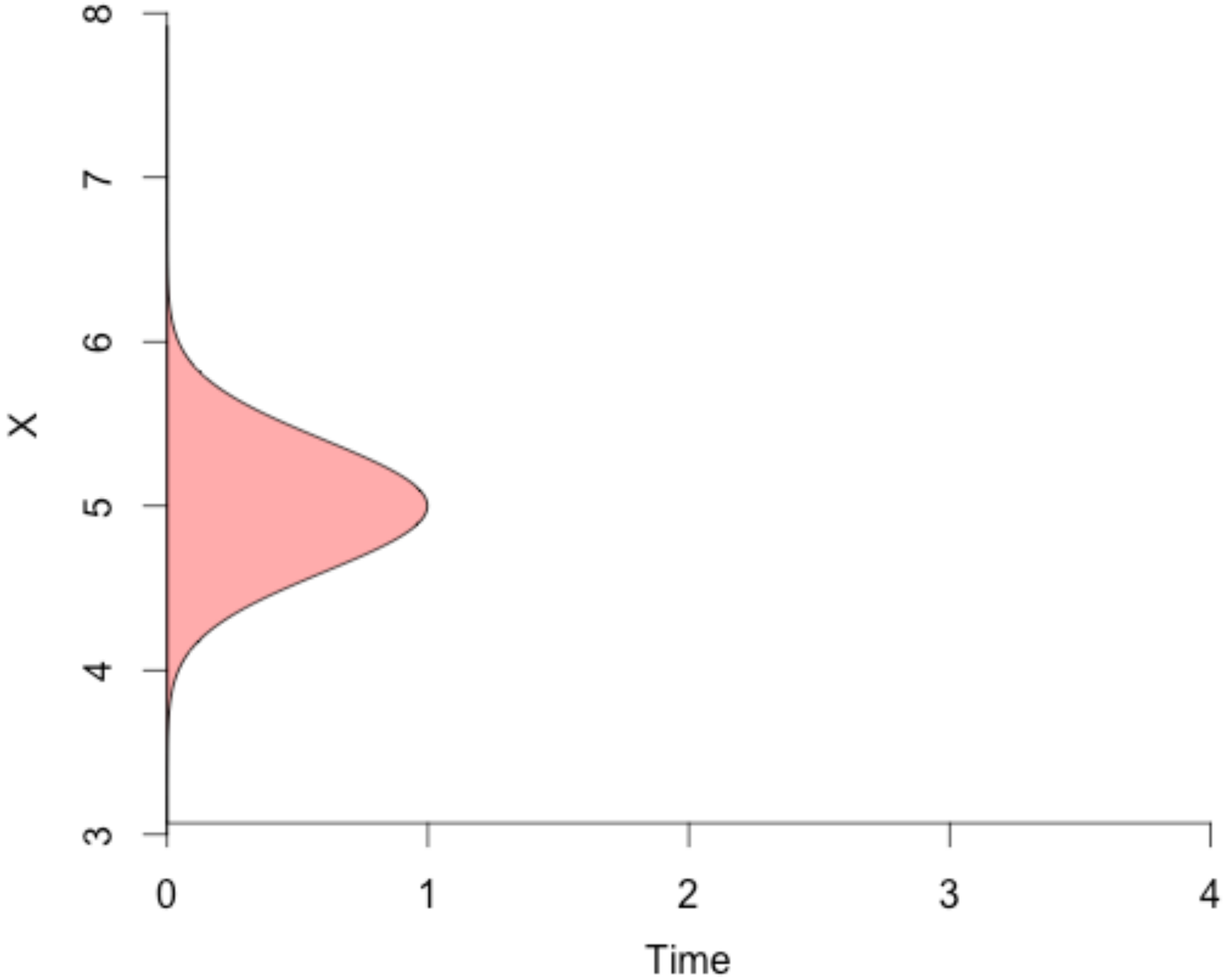
Current State | Future state | Observations

The Analysis Problem

- Prior to observing how the future plays out, what is our best estimate of the future state of the system, X_{t+1}?
  - The forecast, $P(X_{t+1})$

- Once we make (imperfect) observations of the system, Y_t, what's our best estimate of X_t?
  - $P(X_{t+1}) = P(Y_{t+1})$?
  - $P(X_{t+1} \mid Y_{t+1}) \propto P(Y_{t+1} \mid X_{t+1})\, P(X_{t+1})$

Posterior — Likelihood — Prior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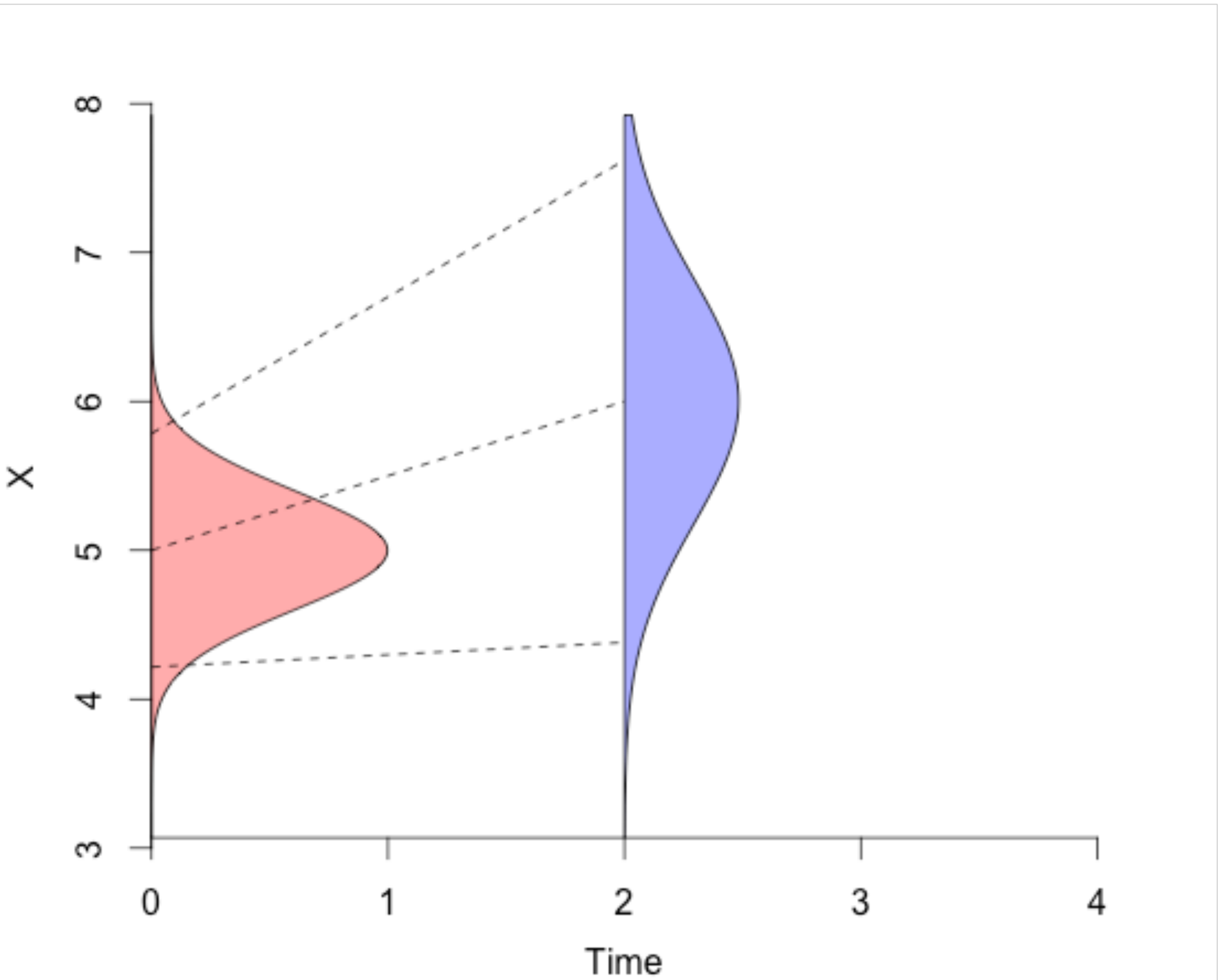
Time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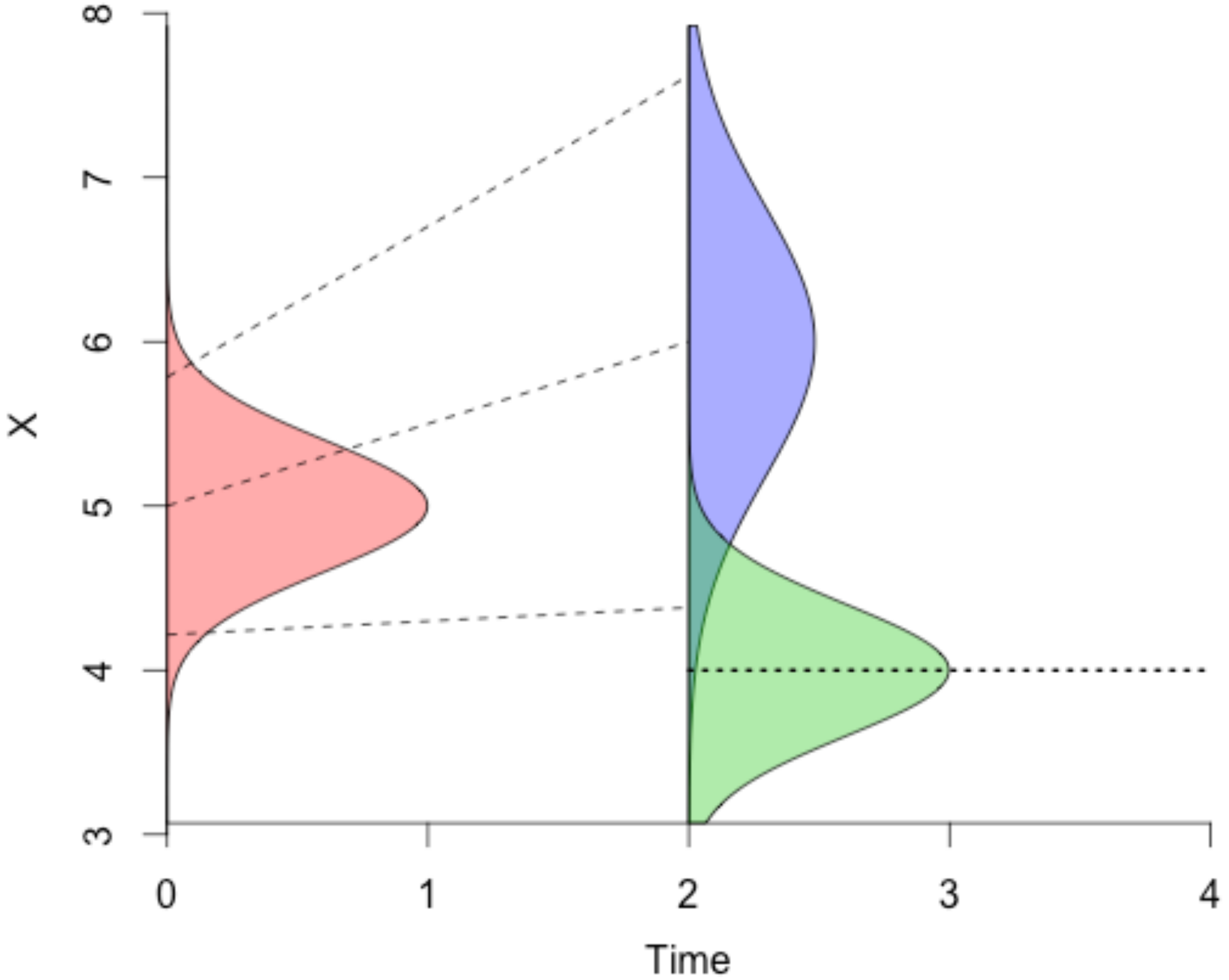
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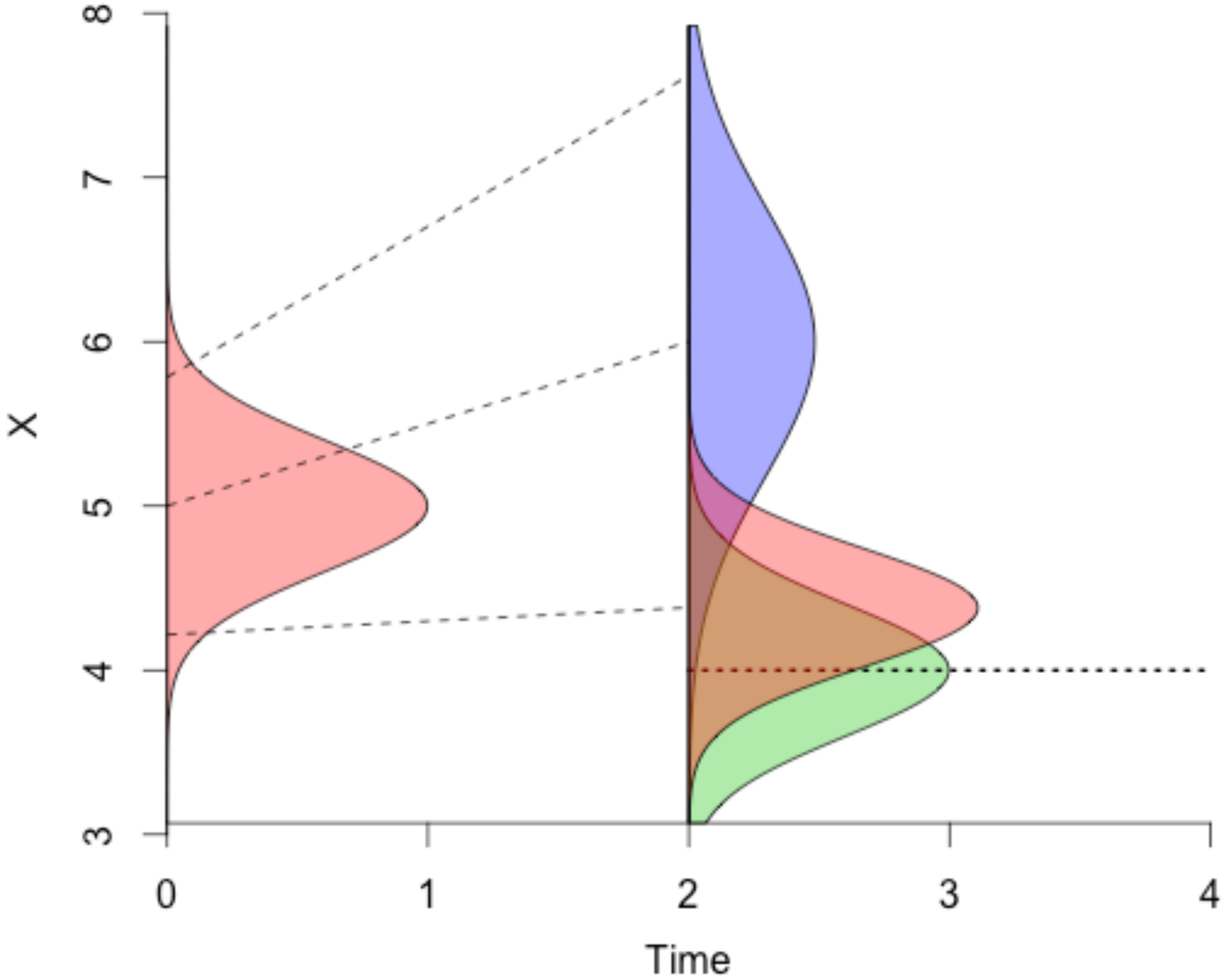
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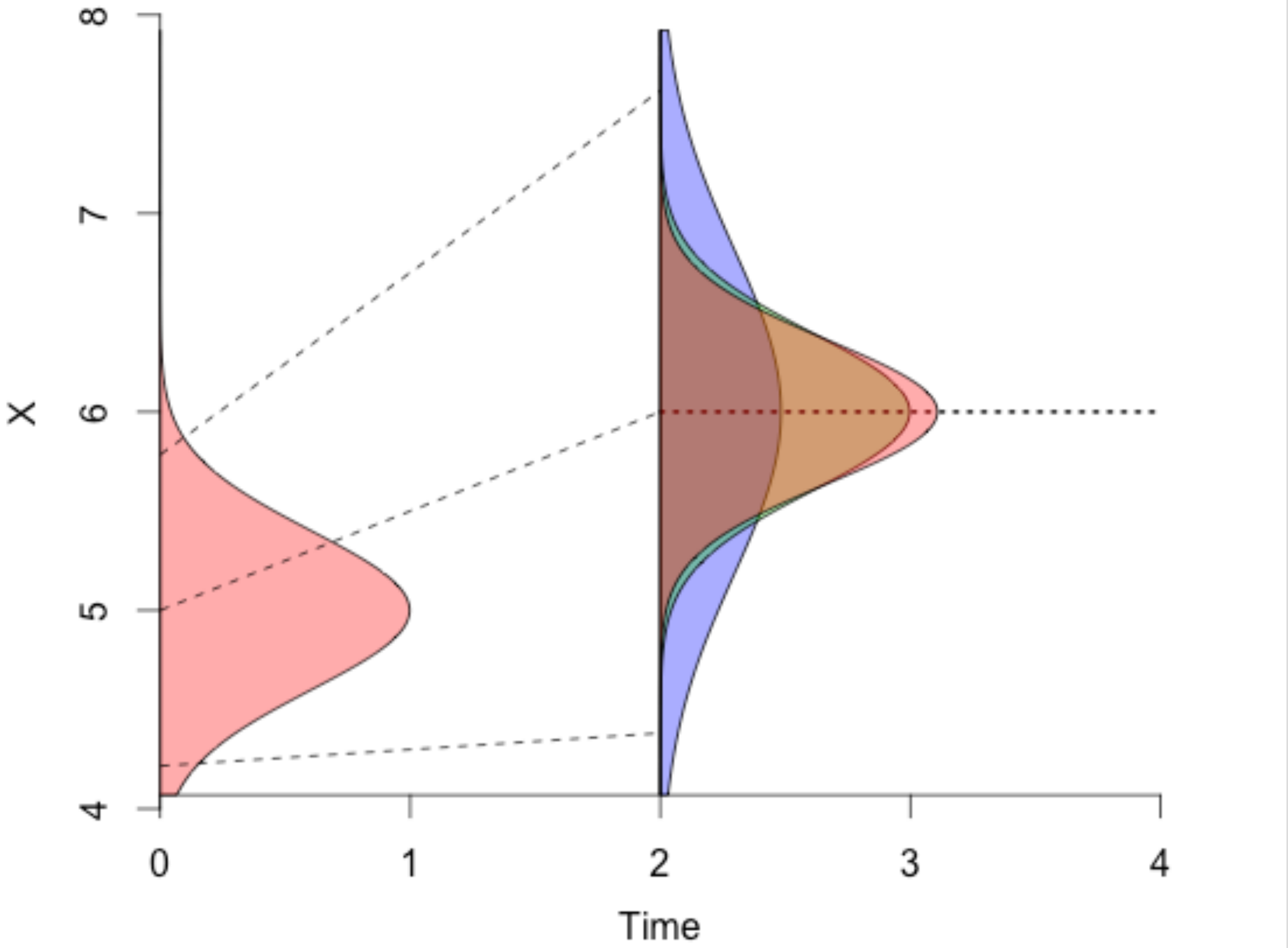

X
8
7
6
5
4
0
1
2
3
4
Time

Simplest Analysis

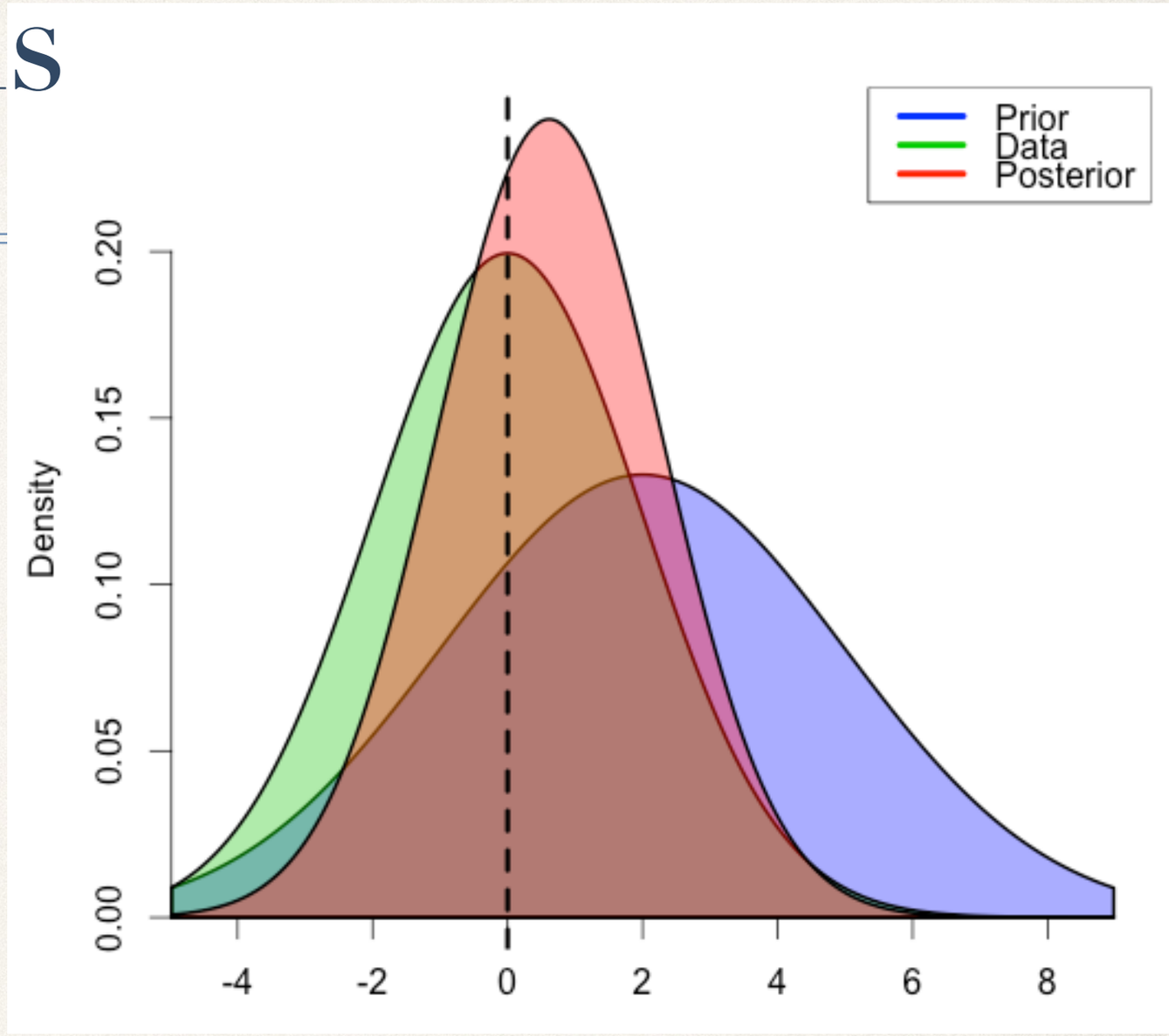
- Forecast:
  Assume $P(X_{t+1}) \sim N(\mu_f, p_f)$

- Observation error:
  Assume $P(Y_{t+1} \mid X_{t+1}) \sim N(X_{t+1}, r)$

  - Likelihood = Data model

- Assume Y, μ_f, p_f and r are known

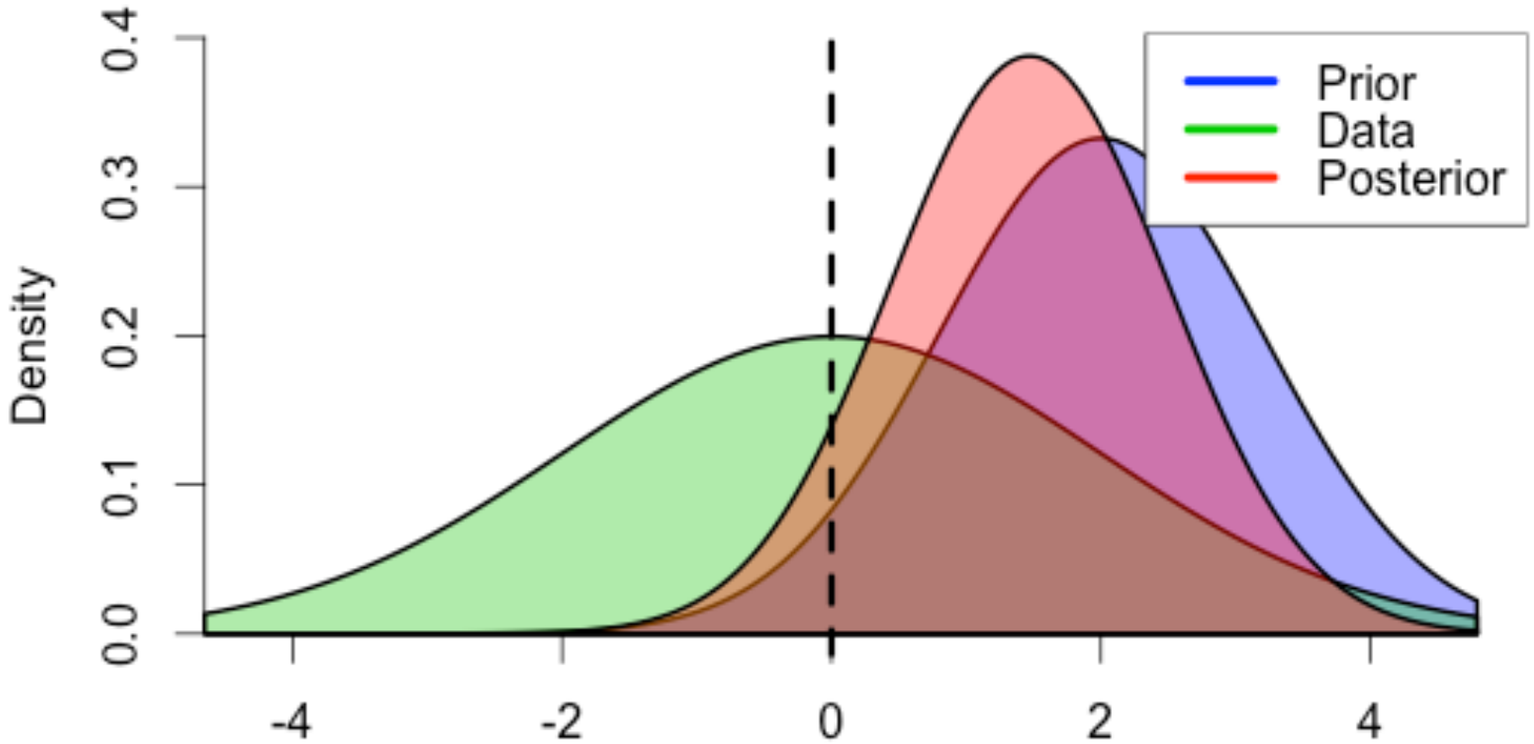
- $P(X_{t+1} \mid Y_{t+1}) \sim N(\mu_a, p_a)$



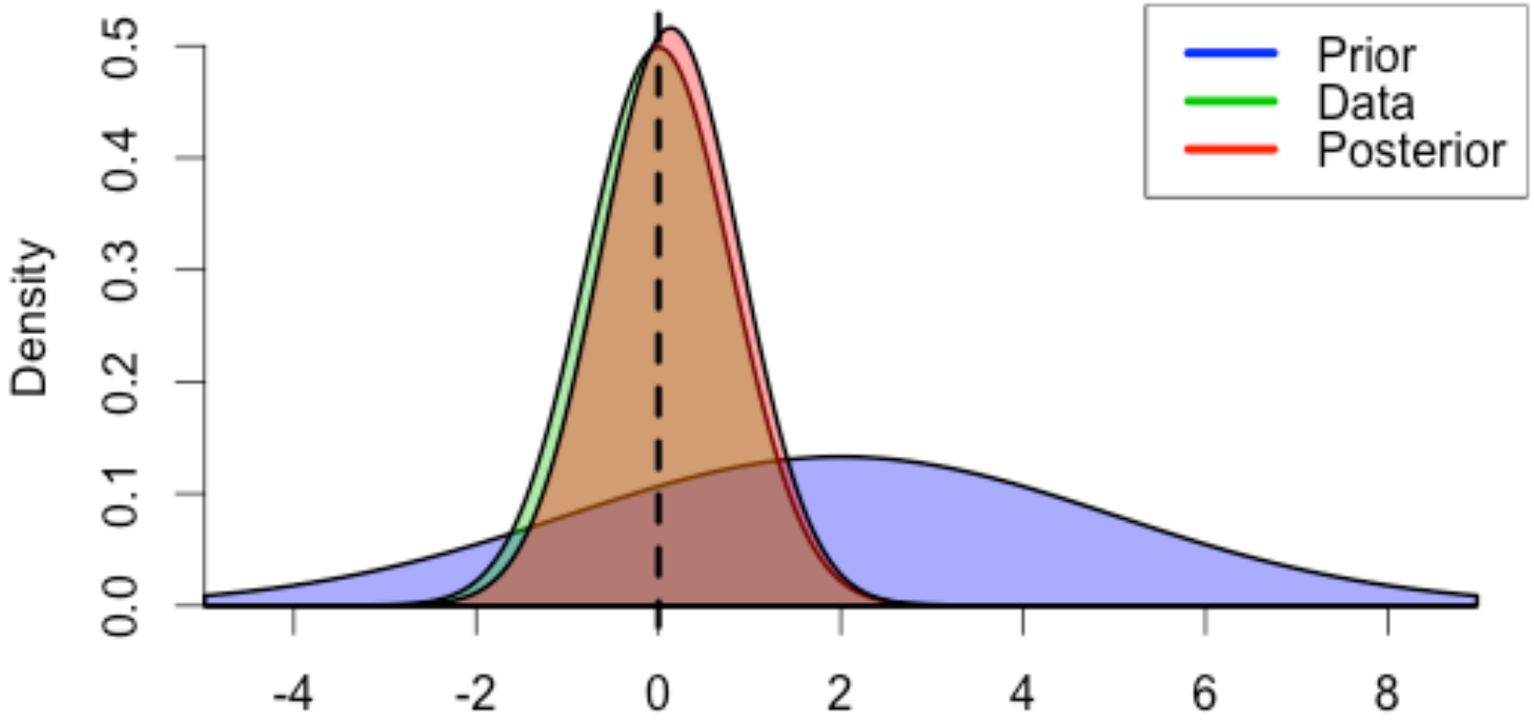
$$\rho = 1/r \qquad \phi = 1/p_f$$

$$X \mid Y \sim N\left(\frac{\rho}{n\rho+\phi} n\bar{Y} + \frac{\phi}{n\rho+\phi}\mu_f, n\rho+\phi\right)$$

Less Precise Data



Less Precise Model



Precision controls influence

Simplest Forecast

- Process Model
  $X_{t+1} = mX_t + \varepsilon_t$
- Process error
  $\varepsilon_t \sim N(0,q)$
- Assume m and q are known
- State uncertainty (IC)
  $P(X_t \mid Y_t) \sim N(\mu_a, p_a)$
- What is $P(X_{t+1})$?

- $E[X_{t+1}] = E[mX_t + \varepsilon_t] = m\mu_a$
- $Var[X_{t+1}] = Var[mX_t + \varepsilon_t]$
  - $m^2Var[X_t] + Var[\varepsilon_t] - 2Cov[mX_t,\varepsilon_t]$
  - $\approx m^2Var[X_t] + Var[\varepsilon_t]$
  - $m^2p_a + q$
- $P(X_{t+1}) \sim N(mX_t, m^2p_a + q)$

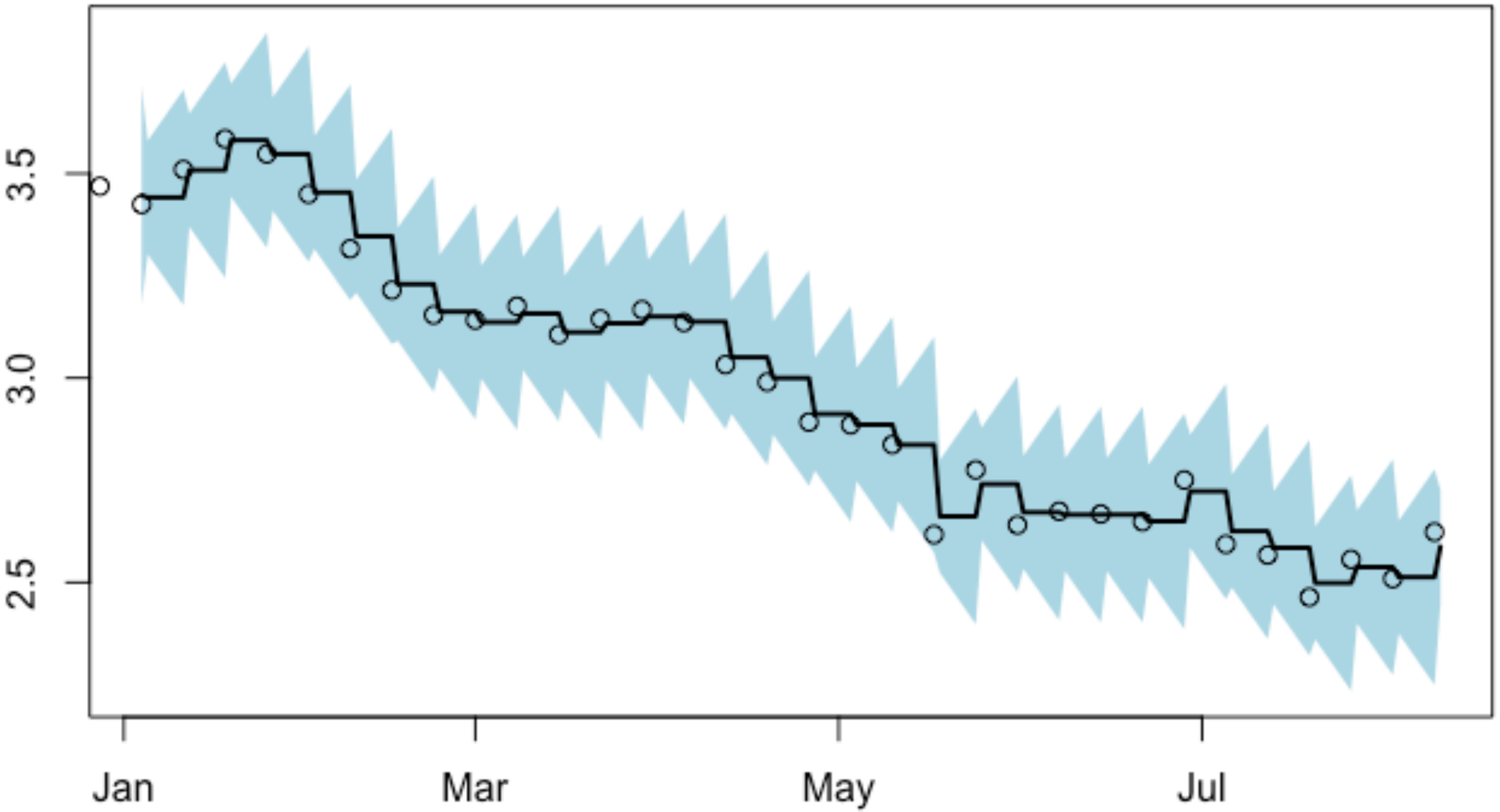
Forecast Cycle

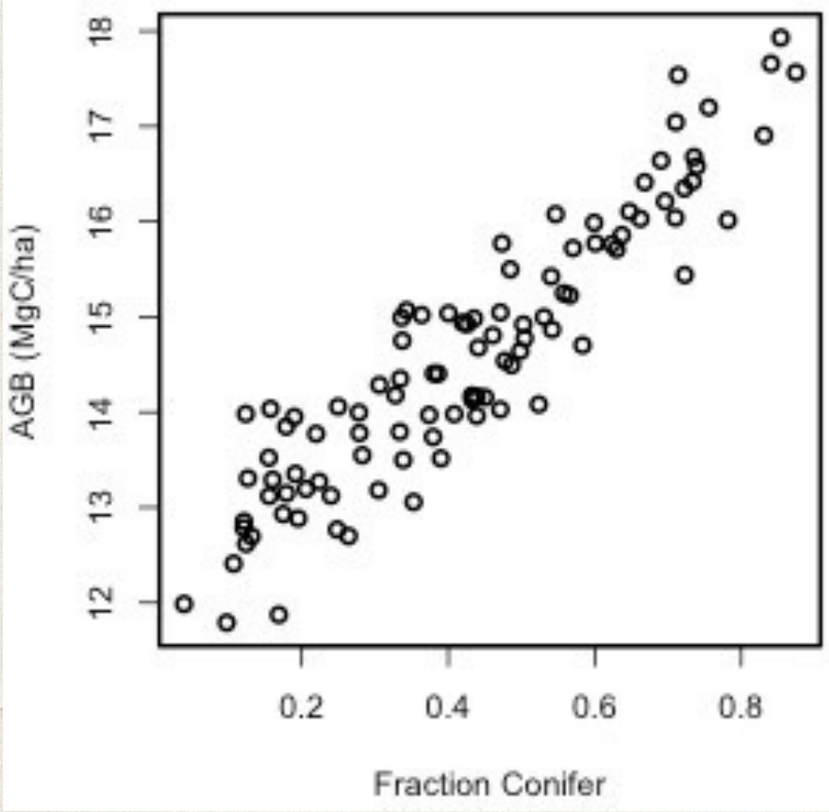
- Forecast Step:
  $P(X_{t+1}) \sim N(\mu_f = mX_t,$
  $p_f = m^2 p_a + q)$
- Analysis Step
  $P(X_{t+1} \mid Y_{t+1}) \sim N(\mu_a, p_a)$
  - $1/p_a = n/p_f + 1/r$
  - $\mu_a = (\mu_f/p_f + nY/r) \cdot p_a$
- Has an analytical solution!
- Kalman Filter

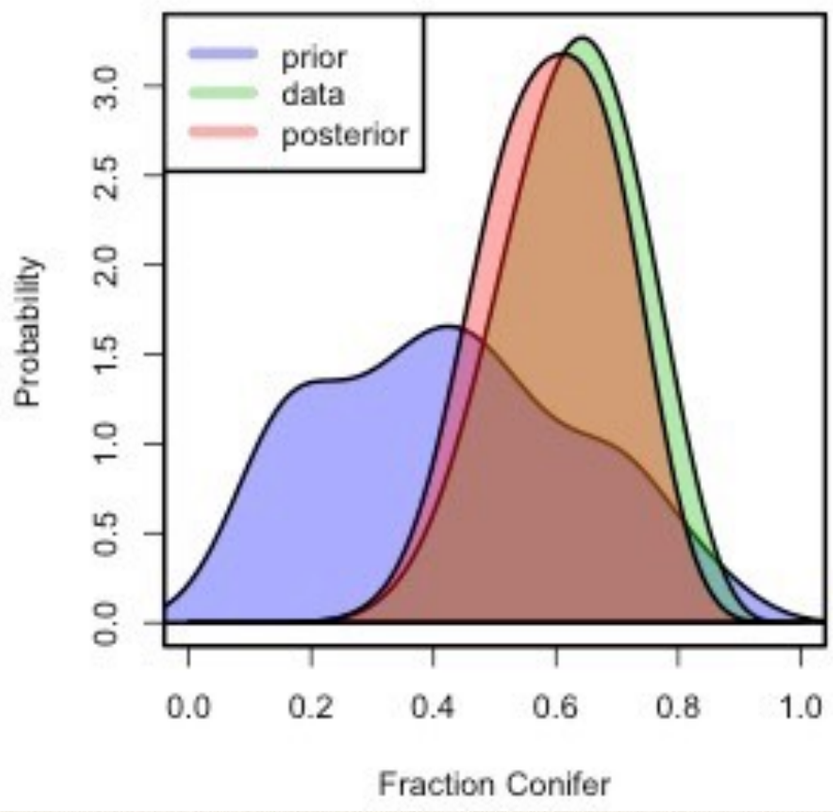
Rudolf Kalman

"Data assimilation isn't rocket science,
but you can use it for that."

– DAVE MOORE


3.5
3.0
2.5
Jan
Mar
May
Jul


AGB (MgC/ha)
12
13
14
15
16
17
18
0.2
0.4
0.6
0.8
Fraction Conifer


prior
data
posterior
Probability
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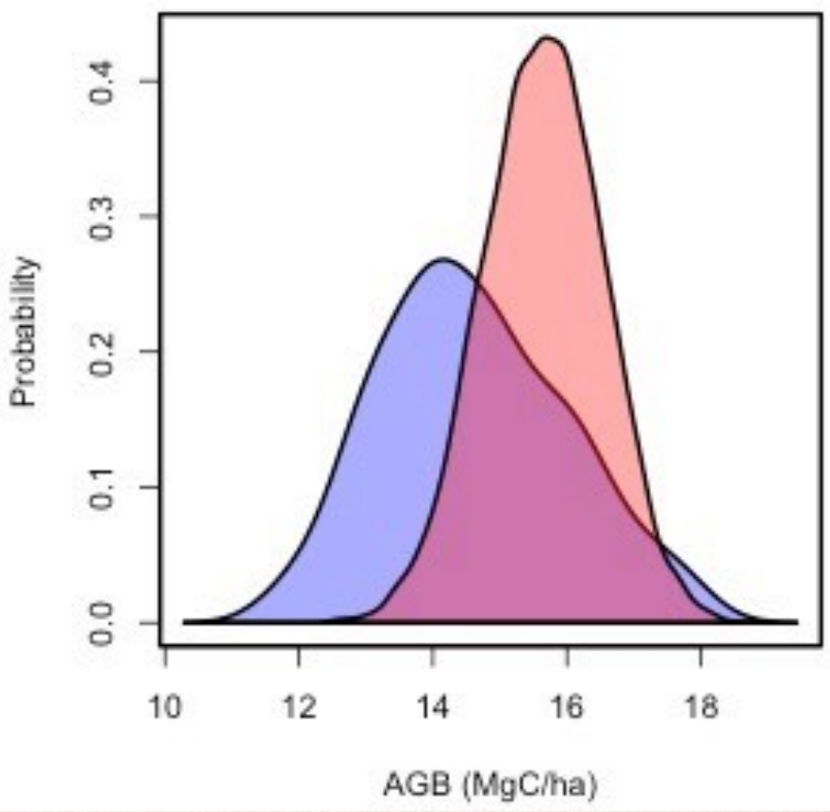
0.5
1.0
1.5
2.0
2.5
3.0
0.0
0.2
0.4
0.6
0.8
1.0
Fraction Conifer


Probability
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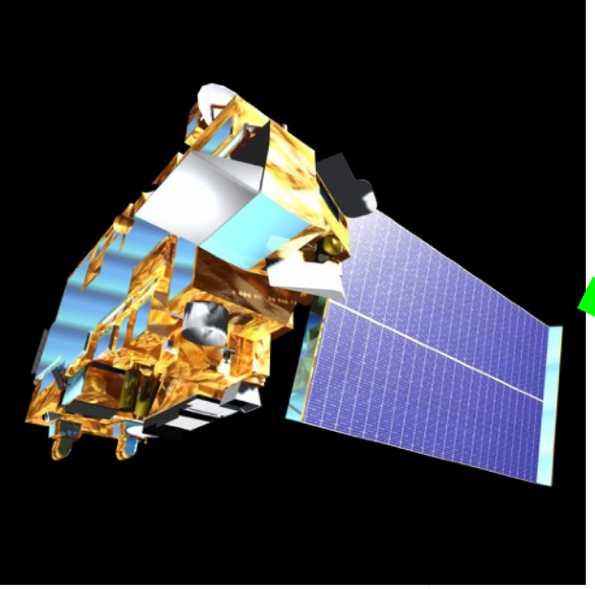
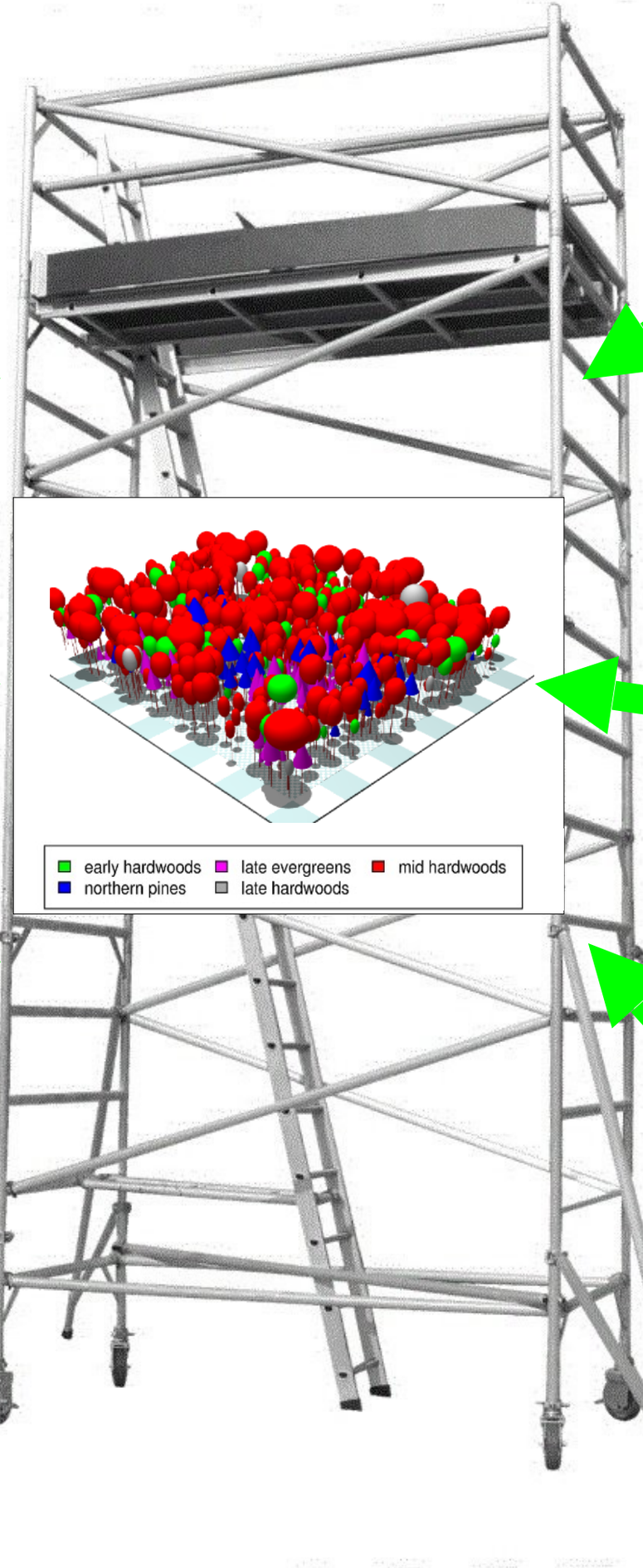
AGB (MgC/ha)




early hardwoods
late evergreens
mid hardwoods
northern pines
late hardwoods

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Generalized to Multivariate

- (n x 1) vector of state means, μ_a or μ_f
- (n x n) state error covariance matrix, P_a or P_f (was p_a, p_f)
- (p x 1) vector of observations, Y
- (p x p) observation error covariance matrix, R (was r)
- (p x n) observation matrix, H
- (n x n) linear process model, M (was m)
- (n x n) process error covariance matrix, Q (was q)

$$X_a|Y \sim N(Y|HX_a, R)\, N(X_a|\mu_f, P_f)$$

$$X_a|Y \sim N(Y|HX_a, R)\, N(X_a|\mu_f, P_f)$$

- Solves to be

$$X_a|Y \sim N\Big(\big(H^T R^{-1} H + P_f^{-1}\big)^{-1}\big(H^T R^{-1} Y + P_f^{-1}\mu_f\big),$$
$$\big(H^T R^{-1} H + P_f^{-1}\big)^{-1}\Big)$$

$$P_a^{-1} = H^T R^{-1} H + P_f^{-1}$$

$$X_a|Y \sim N(Y|HX_a, R)\, N(X_a|\mu_f, P_f)$$

- Solves to be

$$X_a|Y \sim N\Big(\big(H^T R^{-1} H + P_f^{-1}\big)^{-1}\big(H^T R^{-1} Y + P_f^{-1}\mu_f\big), \big(H^T R^{-1} H + P_f^{-1}\big)^{-1}\Big)$$

- Mean and variance simplify to

$$E[X_a|Y] = \mu_a = \mu_f + K\big(Y - H\mu_f\big)$$

$$Var[X_a|Y] = P_a = \big(I - KH\big)P_f$$

$$K = P_{fH}^T\big(R + H P_f H^T\big)^{-1}$$ **Kalman Gain**

Example

- Assume $\mu_f = \{\mu_1,\mu_2,\mu_3\}$, $Y = \{y_2,y_3\}$, and observation error is $R = \sigma^2 I$

$$H = \begin{matrix} X_1 & X_2 & X_3 \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{matrix} Y_2 \\ Y_3 \end{matrix} \end{matrix}$$

- The posterior mean for the unobserved X1 is

$$E[x_1] = \mu_1 + w_{12}(y_2 - \mu_2) + w_{13}(y_3 - \mu_3)$$

$$w_1^T = (p_{12} \quad p_{13}) \begin{pmatrix} p_{22}+\sigma^2 & p_{23} \\ p_{32} & p_{33}+\sigma^2 \end{pmatrix}^{-1}$$

If X's are locations and P is a spatial covariance matrix, model is equivalent to Kriging

covariance between knowns and unknown

covariances among things we know

Forecast Step

$$X_{t+1} = MX_t + \epsilon$$

The posterior distribution of X_{t+1} given X_t is multivariate normal with

$$\mu_{f,t+1} = E[X_{f,t+1} | X_{a,t}] = M_t \mu_{a,t}$$

$$P_{f,t+1} = Var[X_{f,t+1} | X_{a,t}] = Q_t + M_t P_{a,t-1} M_t^T$$

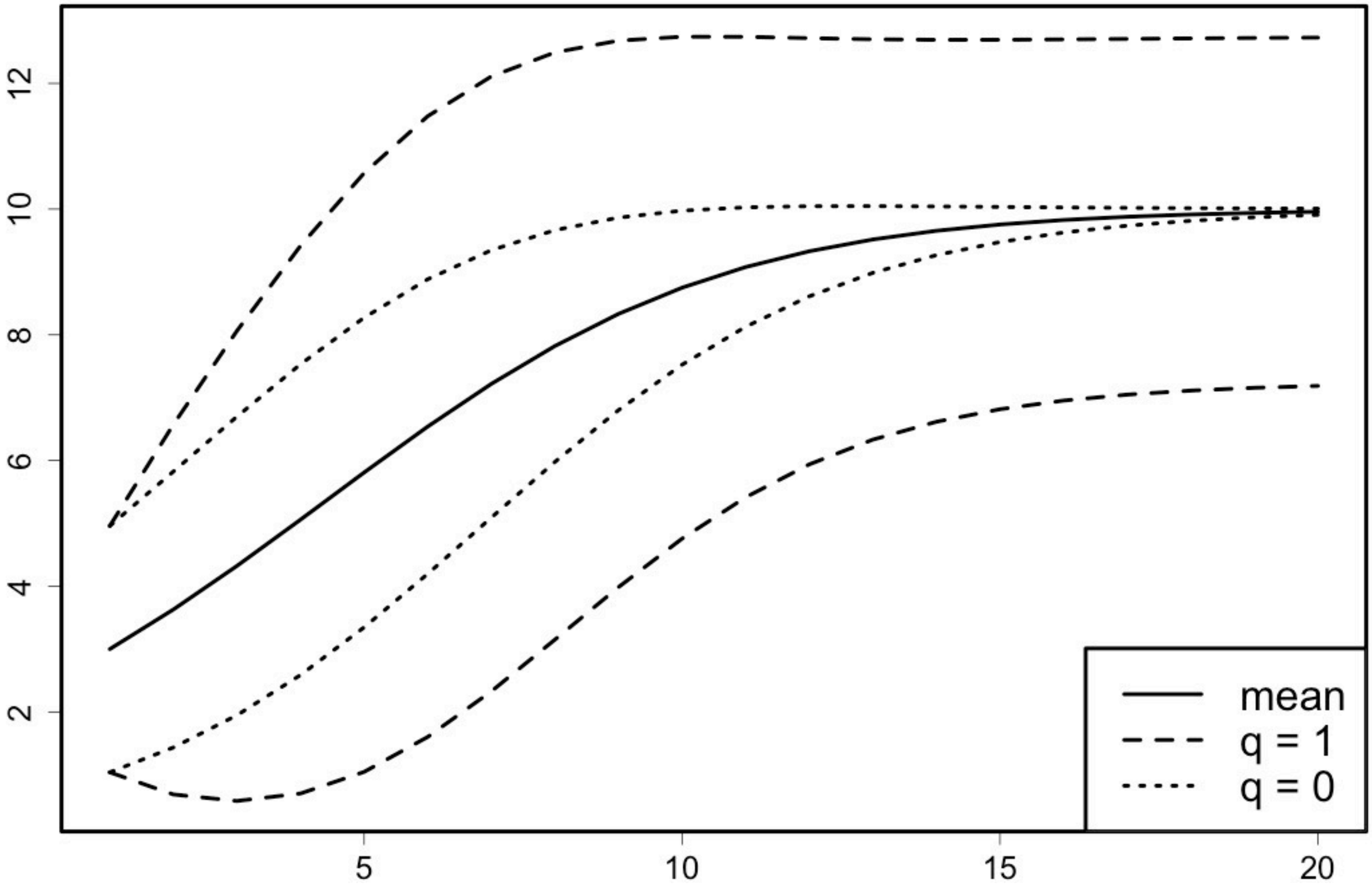
Pro/Con of Kalman Filter (KF)

- Analytically tractable
- Depends only upon the PREVIOUS state, the current Forecast, and the current Data
- Linear
- Normal
- Matrix inversion
- Assumes all parameters (H, R, M, Q) are **known**
- Forward only

UNCERTAINTY PROPAGATION
APPLIED IN THE FORECAST STEP

Approach		Output: Distribution	Output: Moments	
	Analytic	Variable Transform	Analytical Moments Taylor Series	KF EKF
	Numeric	Monte Carlo PF	Ensemble	EnKF

Extended Kalman Filter (EKF)

- Addresses **linear** assumption of the Forecast
  - $\mu_f = f(\mu_a)$
- Update variance using a Taylor Series expansion
  - F = Jacobian (df_i / dx_j)
  - $P_f \approx Q + F\, P_a\, F^T$ (was $Q + M\, P_a\, M^T$)
  - Can be extended to higher orders
- Jensen's Inequality: Biased, Normality assumption FALSE

$$N_{t+1} = N_t + rN_t\left(1 + \frac{N}{K}\right) \quad \rightarrow \quad \frac{\partial N_{t+1}}{\partial N_t} = 1 + r - \frac{2r}{K}N_t$$